ON THE BUCKLING UNDER TENSION OF A MEMBRANE CONTAINING HOLES

(O VYPUCHIVANII MEMBRAN S OTVERSTIIAMI PRI RASTIAZHENII)

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When thin elastic plates containing holes are under tension and in a state of plane stress, generally speaking there arise regions of compressive stress near the holes. The compressive stresses may attain such a magnitude that in the regions where they are acting the plate becomes unstable and buckles, whereupon the state of stress in the remaining unbuckled portion of the plate is drastically changed. The buckling of regions near holes in thin plates under tension has been well documented by experiment. The analysis of the buckling under tension of a thin plate containing holes is necessary for a correct calculation of the stress concentration near the holes. Hence considerable interest attaches to the formulation and solution of the problem of buckling of a thin plate with holes under tension.

Taken as a whole, the problem of buckling for a plate with a finite flexural rigidity at this time presents apparently insurmountable difficulties of a mathematical and even more fundamental nature.

However, in the case of a plate with zero flexural rigidity, the buckling problem allows an exact mathematical formulation and may be effectively solved for a large number of technically important problems.

Some problems of the buckling under tension of a plane membrane containing holes are formulated and solved below. As usual, by a membrane we mean a plate with flexural rigidity equal to zero. The mathematical problem reduces to the solution of a certain quasilinear system of equations in the first partial derivatives, which is of the parabolic type, for the buckled region, and the classical equations of the plane problem of elasticity theory in the unbuckled region. The boundary of the buckled region is unknown beforehand and must be determined in the process of solving the problem. The solution sought here is the asymptotic limit of the solution for a plate with a nonzero flexural rigidity. The other asymptotic limit, in which the flexural rigidity of the plate becomes infinitely large, is given by the solution of the plane problem of elasticity theory for the exterior of a hole. The investigation of the buckling of a membrane with holes under tension is by itself of interest, since there are a large number of flexible plates and films which closely approach a membrane.

The problem of buckling of a membrane turns out to be closely related to the problem of failure under compression of an elastic body, the tensile strength of which is much less than the compressive strength. It is shown that the mathematical formulations of these classes of problems are equivalent, so that the solution of the problem of failure of certain materials may be modelled on a membrane.

In the paper there is also indicated a means of approximately including the flexural rigidity of the plate (in problems of fracture, corresponding to the tensile strength of the elastic body).

1. Formulation of the problem. 1. We imagine a membrane in a state of plane stress and having a buckled region S. We assume that the displacements almost everywhere on the contour L of the buckled region S are small in comparison with a characteristic dimension of the region S. The unbuckled region of the membrane, being in a state of plane stress, fulfils the conditions of the plane problem of linear elasticity theory, while the displacements in the region S normal to the plane of the unbuckled region will be assumed small in comparison with the characteristic dimension of the region S. Hence it is possible to assume that the state of stress in region S is described by the averages over the thickness of the membrane of the stresses σ_x, σ_y and τ_{xy} (the plane of Cartesian coordinates xy coincides with the plane of the unbuckled portion of the membrane). These stresses satisfy the equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1.1}$$

The principal stresses σ_1 and σ_2, of which σ_1 is the minimum and σ_2 the maximum, are determined by the well-known formula

$$\sigma_{2,1} = \tfrac{1}{2}(\sigma_x + \sigma_y) \mp \tfrac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2} \tag{1.2}$$

In a membrane it is not possible to have negative (compressive) principal stresses [1]. Hence a necessary and sufficient indication of the buckled region is the vanishing of the minimum principal stress σ_1. Therefore from formula (1.2) we obtain a relation between the stresses which must be identically satisfied in the buckled region S

$$\sigma_x \sigma_y = \tau_{xy}^2 \qquad (\sigma_x \geqslant 0, \quad \sigma_y \geqslant 0) \tag{1.3}$$

If the stresses are given on the portion of the contour L which coincides with the boundaries of the membrane, then the relations (1.1) and (1.3) form a complete system of equations for the determination of the state of stress in the buckled region S, which is moreover independent of the deformation (a statically determinate problem).

We will assume that on the boundary of the buckled region S there are no discontinuities in the displacements. Then clearly all stress components will be continuous as one passes from the region S into the unbuckled region.

The stresses in the unbuckled region of the membrane are described [2] by the potentials $\Phi(z)$ and $\Psi(z)$ of Kolosov-Muskhelishvili

$$\begin{aligned} \sigma_x + \sigma_y &= 2\,[\Phi(z) + \overline{\Phi(z)}] \qquad (z = x + iy) \\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 2\,[\bar{z}\Phi'(z) + \Psi(z)] \end{aligned} \tag{1.4}$$

2. Let us consider a somewhat different problem. An elastic body with holes, which is in a state describable by the plane problem of elasticity theory (a state of plane strain or plane stress), is subjected to compression. We assume that the tensile strength of the material is much smaller than its compressive strength, while the applied loads are large in comparison with the tensile strength of the material. Such properties are possessed by a large number of real materials, for example, glass, cast iron, soft soil, and so forth. Under compression of the body there arise near the holes regions of tensile stress. Since the material cannot withstand tensile stresses, fracture zones appear near the holes. For the materials mentioned just now it may be assumed that in the fracture zone the maximum principal stress σ_2 vanishes, whence from formula (1.2) we obtain the relation (1.3) between the stresses in the fracture zone (however, in this case we have $\sigma_x \leqslant 0$, $\sigma_y \leqslant 0$).

In the fracture region the equilibrium equations (1.1) are also valid. Thus the mathematical formulation of the present problem of fracture coincides with the problem of buckling of a membrane under tension. This enables one to solve the fracture problem with the aid of the corresponding mathematical problem modelled on the membrane.

In the following, for the sake of definiteness, we will refer everywhere to the buckling of a membrane. Obviously, the solution of the

fracture problem is obtained by changing the sign of the stress in the corresponding buckling problem.

2. Stresses in the buckled region. 1. We will consider equations (1.1) and (1.3). Condition (1.3) is satisfied if we take

$$\sigma_x = \alpha^2 (x, y), \qquad \sigma_y = \beta^2 (x, y), \qquad \tau_{xy} = \alpha (x, y) \beta (x, y) \qquad (2.1)$$

Introducing these values into the equations of equilibrium, we obtain a system of two quasilinear differential equations in the first partial derivatives of the unknown functions $\alpha(x, y)$ and $\beta(x, y)$

$$2\alpha \frac{\partial \alpha}{\partial x} + \beta \frac{\partial \alpha}{\partial y} + \alpha \frac{\partial \beta}{\partial y} = 0, \qquad \beta \frac{\partial \alpha}{\partial x} + \alpha \frac{\partial \beta}{\partial x} + 2\beta \frac{\partial \beta}{\partial y} = 0 \qquad (2.2)$$

We seek the characteristics of the system (2.2). For this as usual [3] we supplement equations (2.2) by the two conditions

$$\frac{\partial \alpha}{\partial x} dx + \frac{\partial \alpha}{\partial y} dy = d\alpha, \qquad \frac{\partial \beta}{dx} dx + \frac{\partial \beta}{\partial y} dy = d\beta$$

and we seek the curves for which the Cauchy problem is insolvable. For such curves the determinant of the system of linear algebraic equations in the partial derivatives of the functions $\alpha(x, y)$ and $\beta(x, y)$ reduces to zero.

We obtain

$$\alpha dy - \beta dx = 0 \qquad (2.3)$$

We require that this curve be the curve of a weak discontinuity. For this it is necessary that the numerator in Cramer's formula vanish, whence we find the condition which must be satisfied on the characteristic curves

$$\beta d\alpha - \alpha d\beta = 0 \qquad (2.4)$$

Thus the system (2.2) has one family of characteristics, which are the straight lines $y = Cx + C_0$, along which the relation $\beta = C\alpha$ is satisfied (system of parabolic type).

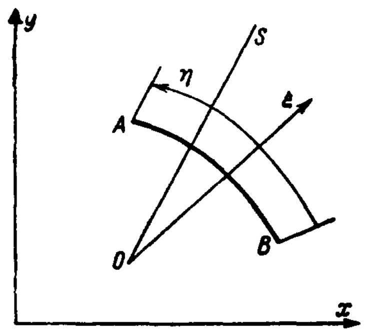
We note that the equilibrium equations and the condition (1.3) are invariant under a change of coordinates from the xy system to any other system of Cartesian coordinates $\xi\eta$. If the ξ-axis is chosen along the characteristic, while the η-axis corresponds to the normal to it, then we find that along the characteristic $\eta = 0$ the following condition is satisfied

$$\sigma_\eta = \tau_{\xi\eta} = 0 \qquad \text{along } \eta = 0 \qquad (2.5)$$

From the formulas (2.5) follows a simple mechanical interpretation of the characteristic: the characteristic is the curve along which buckling of the membrane occurs.

2. We consider the solution of the Cauchy problem for the parabolic system of equations (1.1) and (1.3). Let there be given a smooth curve AB in the xy plane (see the figure) described by $x = x_0(s)$, $y = y_0(s)$, on which are specified the bounded and continuous functions $\sigma_n = \sigma_n(s)$, $\tau_{nt} = \tau_{nt}(s)$ representing given loads.

The Cartesian coordinate axes tn are formed, respectively, by the tangential and normal directions to the arc AB at any point. We assume for the present that the functions $\sigma_n(s)$ and $\tau_{nt}(s)$ nowhere vanish simultaneously on the arc AB. From any point of the arc AB we construct a straight-line characteristic, the equation of which is written in the form



$$x - x_0(s) = -f(s)[y - y_0(s)]$$

$$f(s) = \frac{\tau_{nt}(s)\, x_0'(s) + \sigma_n(s)\, y_0'(s)}{\tau_{nt}(s)\, y_0'(s) - \sigma_n(s)\, x_0'(s)} \qquad (2.6)$$

The lines (2.6) form a one-parameter family in s. We denote the region bounded by the arc AB and the straight characteristics emanating from the ends A and B by S. Since $f(s)$ is a continuous function of s, then through each point of the region S there passes clearly at least one rectilinear characteristic. We will prove the following fundamental theorem:

Theorem 2.1. The Cauchy problem for the parabolic system of equations (1.1) and (1.3) is solvable if and only if the function $f(s)$ is monotonic in the broad sense. If this condition is satisfied, the solution of the Cauchy problem exists and is unique in the entire region S; the solution is found here in closed form.

Monotonicity of the function $f(s)$ in the broad sense means geometrically that a bundle of rectilinear characteristics intersecting the arc AB consists of different parallel lines. We assume that the function $f(s)$ is not monotonic. Then there is at least one point of intersection of the characteristics. This point we will call singular. It is clear that at the singular point there may be: (1) either zero stress components $\sigma_x = \sigma_y = \tau_{xy} = 0$; or (2) an infinitely large radial stress and the remaining stresses equal to zero. The second case corresponds to the presence at the singular point of the buckled region of some concentrated force which is specified beforehand.

Let there be no concentrated forces in the buckled region S of the membrane. The singular point is the center from which there spreads out a fan of radial characteristics. The points of intersection of these characteristics with the arc AB are also singular points, which contradicts the initial assumption that the functions $\sigma_n(s)$ and $\tau_{nt}(s)$ do not vanish simultaneously anywhere on the arc AB. Thus the necessity of the monotonicity in the broad sense of the function $f(s)$ for the solvability of the Cauchy problem has been proved. From the preceding considerations there proceeds the following important corollary.

Corollary. If there are no concentrated forces in region S and if there is even one singular point in region S, then every point of region S is singular.

We will construct the solution of the Cauchy problem, assuming that the function $f(s)$ is monotonic in the broad sense.

We consider an orthogonal system of curvilinear coordinates $\xi\eta$ in region S, formed by a bundle of nonintersecting straight characteristics $\xi(x, y)$ and a family of curves $\eta(x, y)$ orthogonal to them (see the figure). The equations of the curves orthogonal to the characteristics are obviously found by solving the equation

$$\frac{dy}{dx} = -\frac{1}{f(s)} \tag{2.7}$$

where s is in turn a function of the independent variables x and y, defined implicitly by equation (2.6).

In particular, when the tangential stress $\tau_{nt}(s)$ is equal to zero on the arc AB, the family of lines $\xi(x, y)$ is normal to the arc AB (which in this case must not be concave if the Cauchy problem is to be solvable), while the family $\eta(x, y)$ consists of the arc AB itself and curves in the region S which are equidistant from AB as measured along the characteristics.

We write the equations of equilibrium (1.1) in orthogonal curvilinear coordinates [4] and assume that $\sigma_\eta = \tau_{\xi\eta} = 0$. One of the equilibrium equations reduces to an identity, while the other takes the form

$$\frac{\partial}{\partial \xi}(H_1 H_2 \sigma_\xi) - \sigma_\xi H_2 \frac{\partial H_1}{\partial \xi} = 0 \tag{2.8}$$

where H_1 and H_2 are the coefficients of Lamé.

In an infinitesimal neighborhood of each rectilinear characteristic the system of orthogonal curvilinear coordinates $\xi\eta$ obviously coincides with the system of polar coordinates $r\theta$ (r is the radius of curvature of the coordinate curve η, θ is the angle), the origin of which is chosen on the extension of the characteristic at the center of curvature. We obtain

$$\frac{\partial H_1}{\partial \xi} = 0, \qquad H_2 = r \tag{2.9}$$

Making use of (2.9), we find the solution of equation (2.8)

$$\sigma_\xi = \frac{\sigma_n^2(s) + \tau_{nt}^2(s)}{\sigma_n(s)} \frac{R(s)}{r} \tag{2.10}$$

Here $R(s)$ is the radius of curvature of the coordinate curve η at the point of intersection of the arc AB and the corresponding characteristic.

In particular, when $\tau_{nt}(s) = 0$ at a certain point, then $R(s)$ is the radius of curvature of the arc AB at that point.

Thus the Cauchy problem for equations (1.1) and (1.3) has been solved in closed form. The uniqueness of the solution of the Cauchy problem follows immediately from the construction of the solution itself.

3. Now let the functions $\sigma_n(s)$ and $\tau_{nt}(s)$ vanish at some point of the arc AB. The characteristic through this point is tangential to the arc AB at this point. On the one side of this tangent towards which the characteristics emanating from the arc AB are directed the theorem just proved is valid. In order to construct the solution of the Cauchy problem on the other side of the tangent it is necessary to be able to solve the Cauchy problem for the arc AB, on which the values of the function are specified and which is either a characteristic or the envelope of the characteristics. We will prove the following theorem:

Theorem 2.2. If the arc AB, on which the functions $\sigma_n(s)$ and $\tau_{nt}(s)$ vanish, is curved, then the solution of the Cauchy problem for equations (1.1) and (1.3) is equal to zero everywhere within the region S. If the arc AB, on which the functions $\sigma_n(s)$ and $\tau_{nt}(s)$ vanish, is a straight line, then the solution of the Cauchy problem is generally speaking not unique and is determined by the formulas

$$\sigma_n = \tau_{nt} = 0, \qquad \sigma_t = f(n) \tag{2.11}$$

where $f(n)$ is an arbitrary nonnegative function.

For a convex arc AB the proof follows immediately as a consequence of the first theorem. Therefore let the arc AB be concave. At least one point in the region S can be found such that the characteristic drawn through it crosses the boundary of the region S. Then according to the corollary to Theorem 2.1 all points of the region S will be singular. In the case where AB is a straight line, the only characteristic which can be constructed through an arbitrary point of the region S and will not intersect the boundary of the region S is a straight line parallel to the boundary of S. The general solution of the problem for a bundle of characteristics parallel to the boundary $n = 0$ is obviously given by the formulas (2.11). We note that according to the foregoing, the solution corresponding to $f(n) \neq 0$ is unstable for small variations in the arc AB and small perturbations in the value of the function on the boundary. Therefore the only acceptable solution for the case where AB is a straight line is also the solution which vanishes in the entire region S.

4. Let the membrane be the exterior of a circle of radius r_0, on which there is applied a constant tensile load $\sigma_r = q$. In this case the characteristics are radii, the buckled region occupies the entire membrane, and the stresses in it are given by the formulas

$$\sigma_\theta = \tau_{r\theta} = 0, \qquad \sigma_r = (qr_0)/r \tag{2.12}$$

In passing to the limit $q \to \infty$, $r_0 \to 0$ such that $\lim(qr_0) = Q$, then we obtain a singular point of the type of a nucleus of strain. The formulas (2.12) also give the solution to the problem of a membrane, the contour of which consists of arcs of circles of radius r_0 with loads $\sigma_r = q$ and circles of radii $r > r_0$ free from stress. The passage to the limit in this case gives a singularity at the origin of the type caused by a concentrated radial force. We note that buckling does not occur if the concentrated force is obtained by taking the limit of a sequence of delta functions representing a normal load on a rectilinear boundary. Therefore in the solution of problems concerning a membrane with concentrated forces it is necessary to show what limiting case they correspond to.

3. Holes with two cusps, completely surrounded by buckled regions. 1. Let there be an infinite elastic membrane containing n holes, the contours of which have up to two cusps and are completely surrounded by zones of buckling. At infinity there are stresses acting which increase according to a polynomial law. Also, on the contours of the holes loads are applied, so that in the buckled zones the stresses are expressed by the formulas

$$\sigma_x + \sigma_y = \sigma_k, \qquad \sigma_y - \sigma_x + 2i\tau_{xy} = 2a \qquad (k = 1, \ldots, n) \tag{3.1}$$

where the constant σ_k may be different in the various buckled zones, and the complex quantity a is the same constant for all the buckled zones.

With the help of the basic relations (1.4) of N.I. Muskhelishvili the boundary conditions on the unknown multiply connected contour L, which divides the buckled and unbuckled regions, may be written in the form

$$4\,\mathrm{Re}\,\Phi(z) = \sigma_k, \qquad \bar{z}\Phi'(z) + \Psi(z) = a \tag{3.2}$$

We pass to the parametral plane of the complex variable ζ with the aid of the transformation $z = \omega(\zeta)$. The analytic function $\omega(\zeta)$ conformally maps any canonical region of the ζ-plane (for example, the exterior of parallel cuts) onto the unbuckled region in the z-plane bounded by the contour L. In the ζ-plane the following boundary value problem is obtained for the three analytic functions $\omega(\zeta)$, $\varphi(\zeta) = \Phi[\omega(\zeta)]$, $\psi(\zeta) = \Psi[\omega(\zeta)]$:

$$4\,\mathrm{Re}\,\varphi(\zeta) = \sigma_k, \qquad \overline{\omega(\zeta)}\,\varphi'(\zeta) + \psi(\zeta)\,\omega'(\zeta) = a\omega'(\zeta) \tag{3.3}$$

We introduce the analytic function $\chi(\zeta)$

$$\chi(\zeta) = \frac{\omega'(\zeta)}{\varphi'(\zeta)}\,[\psi(\zeta) - a] \tag{3.4}$$

The second boundary condition of (3.3) may be easily transformed with the aid of the function $\chi(\zeta)$ into the two Dirichlet problems

$$\mathrm{Re}\,[\omega(\zeta) + \chi(\zeta)] = 0, \qquad \mathrm{Im}\,[\omega(\zeta) - \chi(\zeta)] = 0 \tag{3.5}$$

The Dirichlet problem has been well investigated. In many cases its solution may be found in closed form [5,6].

2. As an example we consider the simplest case of an infinite membrane with a single hole, stretched at infinity by the constant stresses

$$\sigma_y = \sigma_y^{\infty}, \qquad \sigma_x = \sigma_x^{\infty}, \qquad \tau_{xy} = 0 \tag{3.6}$$

The cusps of the contour, which is stress-free, are located on the x-axis at the points $x = \pm\, l$. Further requirements imposed on the contour will be made clear after solving the problem; they arise from the condition that the buckled region surrounds the whole contour of the hole. According to Theorem 2.2, in this case we have $\sigma_k = a = 0$. For the canonical region on the ζ-plane we choose the exterior of the single cut $(-1, +1)$ with the corresponding points $\omega(\pm 1) = \pm l$, $\omega(\infty) = \infty$ in the z-plane.

The boundary conditions for the Dirichlet problems (3.3) and (3.5) for the cut $(-1, +1)$ are written in the form

$$\mathrm{Re}\,\varphi(\zeta) = 0, \qquad \mathrm{Re}\,[\omega(\zeta) + \chi(\zeta)] = 0, \qquad \mathrm{Im}\,[\omega(\zeta) - \chi(\zeta)] = 0 \tag{3.7}$$

According to formulas (1.4), (3.6) and (3.4), for $\zeta \to \infty$ the functions $\varphi(\zeta)$, $\omega(\zeta)$ and $\chi(\zeta)$ behave in the following manner:

$$\varphi(\zeta) = \tfrac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) + O(\zeta^{-2}), \qquad \omega(\zeta) = O(\zeta), \ \chi(\zeta) = O(\zeta^3) \tag{3.8}$$

Moreover, the function $\varphi(\zeta)$ is unbounded, while the functions $\omega(\zeta)$ and $\chi(\zeta)$ are bounded in the neighborhoods of the ends $\zeta = \pm 1$ of the cut. The general solution of the Dirichlet problem (3.7), satisfying the conditions (3.8), is written in the form [5,6]

$$\varphi(\zeta) = \frac{(\sigma_x^\infty + \sigma_y^\infty)\zeta}{4\sqrt{\zeta^2-1}}$$

$$\omega(\zeta) = (-A\zeta^2 - B\zeta + E)\sqrt{\zeta^2-1} + A\zeta^3 + B\zeta^2 + C\zeta + D \tag{3.9}$$

$$\chi(\zeta) = (-A\zeta^2 - B\zeta + E)\sqrt{\zeta^2-1} - A\zeta^3 - B\zeta^2 - C\zeta - D$$

$$(\sqrt{\zeta^2-1} = \zeta + O(\zeta^{-1}) \quad \text{for } \zeta \to \infty)$$

The unknown constants A, B, C, D and E are real. In order to determine them we use the following conditions:

1) condition (3.4), relating certain coefficients in the expansions of the functions $\omega(\zeta)$, $\chi(\zeta)$ and $\varphi(\zeta)$ at infinity;

2) the condition that there be no concentrated forces at the cusps $z = \pm l$ (from which it follows that the singularities of the functions $\Phi(z)$ at these points be integrable);

3) the condition of correspondence of the points $\omega(+1) = l$, $\omega(-1) = -l$. As a result we obtain

$$A = E = 2l\frac{\sigma_y^\infty - \sigma_x^\infty}{\sigma_y^\infty + 3\sigma_x^\infty}, \quad C = l\left(1 - 2\frac{\sigma_y^\infty - \sigma_x^\infty}{\sigma_y^\infty + 3\sigma_x^\infty}\right), \quad B = D = 0 \tag{3.10}$$

We will now find the boundary of the buckled zone. Setting $\zeta = t$ in the mapping function $\omega(\zeta)$, we obtain the equation of its contour

$$x(t) = At^3 + Ct, \qquad y(t) = A(1-t^2)^{3/2} \qquad (1 \geqslant t \geqslant -1) \tag{3.11}$$

The family of curves (3.11) depends on $\sigma_y^\infty/\sigma_x^\infty$, the ratio of the tensile stresses at infinity. All the curves have two axes of symmetry, the x-axis and the y-axis, and are tangent to the x-axis at the points $x = \pm l$. Investigation shows that for $\sigma_y^\infty = \sigma_x^\infty$ the boundary of the buckled zone coincides with the cut $(-l, +l)$ on the real axis; for

$\sigma_x^\infty \leqslant \sigma_y^\infty < 5\sigma_x^\infty$ the contour of the boundary is a smooth curve having a horizontal tangent at $x = 0$. For $\sigma_y^\infty < \sigma_x^\infty$ the buckling occurs on the second sheet of a two-sheet Riemann surface (the sheets are joined along the cut $(-l, +l)$); this case corresponds physically to buckling on pieces of the membrane which are attached to opposite sides of the cut; the contours of the boundary of the buckled region in this case have a horizontal tangent at $x = 0$.

For $\sigma_y^\infty = 5\sigma_x^\infty$ the boundary of the buckled zone will be the astroid (four-cusp hypocycloid) $x^{2/3} + y^{2/3} = l^{2/3}$, having a vertical tangent at $x = 0$, while for $\sigma_y^\infty > 5\sigma_x^\infty$ there appears on the contour of this boundary a loop which has no physical meaning (an analogous circumstance arises in the theory of cracks [7]). Thus for $\sigma_y^\infty \leqslant 5\sigma_x^\infty$, the solution of the formulated problem exists, while for $\sigma_y^\infty > 5\sigma_x^\infty$ the problem in its present formulation is insolvable. Physically this may be explained by the fact that as the cusp appears only on the contour of the zone of buckling, in the neighborhood of this point compressive stresses originate, since the flexural rigidity of the plate is finite, though small. Consequently, the pure membrane formulation of the problem is valid only up to the appearance of a cusp on the contour of the zone of buckling.

Note 3.1. The formulas (3.9) also give the solution of the problem of buckling of a membrane which is cut between two concentrated forces applied at the points $z = \pm l$ and directed opposite of one another, when constant tensile stresses act at infinity. In this case for the determination of the unknown constants the condition (2) should be replaced by the condition that the singularities of the function $\Phi(z)$ at the points $z = \pm l$ are those corresponding to the given forces.

Note 3.2. If the coefficient of intensity of the stress σ_y in the neighborhood of the cusps $z = \pm l$ of an elastic membrane exceeds a certain value, which is a constant for a given material and thickness of the membrane, the length l begins to increase with increasing stress σ_y^∞. The dependence of the length l on the stresses σ_y^∞ and σ_x^∞ and modulus of cohesion K may be determined by using the condition of Barenblatt in the theory of equilibrium cracks [7].

3. In certain cases a finite flexural rigidity of the plate may be taken into account in an approximate manner. We consider the method applied to the simplest example. Let an infinite elastic plate with a cut $(-l, +l)$ along the real axis be subjected to tensile stresses σ_x^∞ and σ_y^∞ at infinity. We assume that in the zone of buckling, which surrounds the whole cut, there are constant stresses (1.1)

$$\sigma_x = \sigma, \qquad \sigma_y = 0, \qquad \tau_{xy} = 0 \tag{3.12}$$

which satisfy the boundary conditions and the equations of equilibrium.

The negative stress σ is of the order of some mean value of the compressive stresses in the buckled region, for which we may use for example the critical stress in a plate of width l (and of the same thickness and material).

The solution of the mathematical problem is carried out in a manner analogous to the previous one. The function $\varphi(\zeta)$ is written in the form

$$\varphi(\zeta) = \frac{\sigma}{4} + \frac{(\sigma_x^{\infty} + \sigma_y^{\infty} - \sigma)\zeta}{4\sqrt{\zeta^2 - 1}} \tag{3.13}$$

The functions $\omega(\zeta)$ and $\chi(\zeta)$ are found by the formulas (3.9), in which it is necessary to assume

$$A = E = 2l\frac{\sigma_y^{\infty} - \sigma_x^{\infty} + \sigma}{\sigma_y^{\infty} + 3\sigma_x^{\infty} - 3\sigma}, \qquad C = l\frac{5\sigma_x^{\infty} - 5\sigma - \sigma_y^{\infty}}{\sigma_y^{\infty} + 3\sigma_x^{\infty} - 3\sigma}, \qquad B = D = 0$$

$$\left(\chi(\zeta) = \frac{\omega'(\zeta)}{\varphi'(\zeta)}\left[\psi(\zeta) + \frac{\sigma}{2}\right]\right) \tag{3.14}$$

The contour of the zone of buckling has the same character as that for the membrane, but the boundary of nonexistence of the solution is moved back considerably. In particular, when $\sigma_x^{\infty} = 0$, buckling begins for $\sigma_y^{\infty} = -\sigma$, and with increasing σ_y^{∞} the region of buckling grows until $\sigma_y^{\infty} \leqslant -5\sigma$ (for $\sigma_y^{\infty} = -5\sigma$ the contour of buckling is an astroid, as before); for $\sigma_y^{\infty} > -5\sigma$ the solution has no physical meaning for the reason mentioned earlier.

It is of interest to find the dependence of the length of a moving equilibrium crack in a plate subjected to loads σ_x^{∞} and σ_y^{∞} and a stress σ in the buckled region. Using the condition of Barenblatt [7] and obtaining the solution, we find

$$\frac{l\sigma^2}{K} = \frac{8\sigma^2(\sigma_y^{\infty} + 3\sigma_x^{\infty} - 3\sigma)}{\pi^2(5\sigma_y^{\infty} - \sigma_x^{\infty} + \sigma)(\sigma_x^{\infty} + \sigma_y^{\infty} - \sigma)^2} \quad \text{(K is the modulus of cohesion)} \tag{3.15}$$

A crack of any length in the plate is always stable for $5\sigma_y^{\infty} \leqslant \sigma_x^{\infty} - \sigma$; for $5\sigma_y^{\infty} > \sigma_x - \sigma$ each value of the crack length corresponds to a certain value of the critical load.

4. Holes partially surrounded by zones of buckling. We will mention one general class of problems of buckling under tension of a membrane with holes, in which a solution may be obtained (sometimes in closed form). We note that a contour free of loads and having no cusps clearly may not be completely surrounded by a region of buckling.

Let the contour of the hole (simply or multiply connected) consist of a certain number of rectilinear segments, on which the normal displacement u_n is specified as a piece-wise constant function, the stress τ_{tn} is equal to zero and arbitrary curvilinear arcs which are completely surrounded by regions of buckling. We assume that in the regions of buckling the following stresses are zero:

$$\sigma_x + \sigma_y = 0, \qquad \sigma_y - \sigma_x + 2i\tau_{xy} = 0 \tag{4.1}$$

The unknown boundary of the buckled and unbuckled zones is denoted by L, while that portion of the boundary of the membrane consisting of the rectilinear segments and lying in the unbuckled region is denoted by M. For an arbitrary Cartesian coordinate system tn the following representation may be used [2]:

$$\begin{gathered}\sigma_t + \sigma_n = 2\,[\Phi(z) + \overline{\Phi(z)}]\\ \sigma_n - \sigma_t + 2i\tau_{tn} = 2e^{2i\theta}\,[\bar{z}\Phi'(z) + \Psi(z)]\\ 2\mu\left(\frac{\partial u_t}{\partial t} + i\frac{\partial u_n}{\partial t}\right) = \varkappa\Phi(z) - \overline{\Phi(z)} - e^{-2i\theta}\,[\overline{z\Phi'(z)} + \overline{\Psi(z)}]\end{gathered} \tag{4.2}$$

Here μ and κ are the elastic constants and θ is the angle between the t-axis and the x-axis. Thus the following expression is obtained for the Kolosov-Muskhelishvili potential $\Phi(z)$

$$(\varkappa + 1)\,\Phi(z) = \sigma_n + 2\mu\frac{\partial u_t}{\partial t} + i\left(-\tau_{tn} + 2\mu\frac{\partial u_n}{\partial t}\right) \tag{4.3}$$

With the help of formulas (4.2) and (4.3), the boundary conditions of the problem may be written in the form

$$\begin{gathered}4\,\mathrm{Re}\,\Phi(z) = 0, \qquad \bar{z}\Phi'(z) + \Psi(z) = 0 \quad \text{on } L\\ \mathrm{Im}\,\Phi(z) = 0, \qquad \mathrm{Im}\,\{e^{2i\theta_j}[\bar{z}\Phi'(z) + \Psi(z)]\} = 0 \quad \text{on } M\end{gathered} \tag{4.4}$$

Here θ_j is the angle formed by the jth rectilinear segment of the boundary with the x-axis (in the direction in which the contour $L + M$ is traversed).

We pass to the parametral plane of the complex variable ζ by means of the transformation $z = \omega(\zeta)$. The analytic function $\omega(\zeta)$ conformally maps the exterior of a cut in the ζ-plane parallel to the real axis onto the unbuckled region of the z-plane bounded by the contour $L + M$. The images of the contours L and M in the ζ-plane will also be denoted by L and M.

In the ζ-plane we obtain from formulas (4.4) the following boundary value problem for the three analytic functions $\omega(\zeta)$, $\varphi(\zeta) = \Phi[\omega(\zeta)]$ and

$\psi(\zeta) = \Psi[\omega(\zeta)]$

$$\begin{gathered}
\operatorname{Im}\varphi(\zeta) = 0 \quad \text{on } M, \qquad 4\operatorname{Re}\varphi(\zeta) = 0 \quad \text{on } L \\
\overline{\omega(\zeta)}\varphi'(\zeta) + \omega'(\zeta)\psi(\zeta) = 0 \quad \text{on } L \\
\operatorname{Im}\left\{e^{2i\theta j}\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\varphi'(\zeta) + \psi(\zeta)\right]\right\} = 0 \quad \text{on } M \\
\operatorname{Im}[e^{-i\theta j}\omega(\zeta)] = d_j \quad \text{on } M
\end{gathered} \tag{4.5}$$

The last condition of (4.5) is the complex form of the equation of the jth rectilinear segment of the boundary

$$y = x \tan\theta_j + d_j / \cos\theta_j$$

We introduce the analytic function $\chi(\zeta)$

$$\chi(\zeta) = \frac{\omega'(\zeta)}{\varphi'(\zeta)}\psi(\zeta) \tag{4.6}$$

With the aid of the function $\chi(\zeta)$ the last three conditions of the boundary value problem (4.5) may be written in the form

$$\overline{\omega(\zeta)} + \chi(\zeta) = 0 \quad \text{on } L \tag{4.7}$$

$$\operatorname{Im}[e^{-i\theta j}\omega(\zeta)] = d_j, \qquad \operatorname{Im}[e^{i\theta j}\chi(\zeta)] = d_j \quad \text{on } M$$

The boundary value problem (4.7) may in certain cases be solved in closed form. For example, let the contour $L + M$ be the totality of segments of the real axis. This may be the case when the contour $L + M$ has an axis of symmetry in the z-plane [8]. We cut the ζ-plane along the real axis and consider the upper half plane $\operatorname{Im}\zeta \gg 0$.

On the cut portions N of the real axis ζ the boundary condition may clearly be given in the form

$$\operatorname{Im}\omega(\zeta) = 0, \qquad \operatorname{Im}\chi(\zeta) = 0 \quad \text{on } N \tag{4.8}$$

The functions $\omega(\zeta)$ and $\chi(\zeta)$ are analytically continued across the segment L of the real axis ζ into the lower half-plane $\operatorname{Im}\zeta < 0$ with the help of the relations

$$\omega(\zeta) + \bar{\chi}(\zeta) = 0, \qquad \bar{\omega}(\zeta) + \chi(\zeta) = 0 \tag{4.9}$$

For the common analytic functions $\omega(\zeta)$ and $\chi(\zeta)$ in the exterior of the cut $M + N$ we obtain on the basis of (4.7) to (4.9) the following linear boundary value problem of Riemann for two functions:

$$\begin{aligned} &\omega^+(\zeta) = -e^{2i\theta_j}\chi^-(\zeta) + 2d_j e^{i\theta_j} \text{ on } M, \qquad \omega^+(\zeta) = -\chi^-(\zeta) \quad \text{on } N \\ &\chi^+(\zeta) = -e^{-2i\theta_j}\omega^-(\zeta) + 2d_j e^{-i\theta_j} \text{ on } M, \qquad \chi^+(\zeta) = -\omega^-(\zeta) \quad \text{on } N \end{aligned} \tag{4.10}$$

The Riemann problem (4.10) is a particular case of a more general Riemann boundary value problem, the solution of which was obtained in [9]. Its solution may be found in closed form.

It is not difficult to show that when the unbuckled region of the z-plane bounded by the contour $L + M$ is simply connected, the problem may always be solved in closed form by a mapping onto the upper half-plane ζ, reducing it exactly as before to the boundary value problem (4.7) and thereupon to the Riemann problem (4.10). Similarly, the solution may be found in the case where the boundary conditions

$$\sigma_n = 0, \quad \frac{\partial u_t}{\partial t} = e_k \qquad \text{or} \qquad \tau_{nt} = 0, \quad \frac{\partial u_n}{\partial t} = f_k$$

are specified on the contour M, where e_k and f_k are arbitrary piecewise constant functions.

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